Lesson 12: The Graph of the Equation $y = f(x)$

Classwork

In Module 1, you graphed equations such as $4x + y = 10$ by plotting the points on the Cartesian coordinate plane that corresponded to all of the ordered pairs of numbers (x, y) that were in the solution set. We called the geometric figure that resulted from plotting those points in the plane the *graph of the equation in two variables*.

In this lesson, we extend this notion of the graph of an equation to the graph of $y = f(x)$ for a function f. In doing so, we use computer thought code to describe the process of generating the ordered pairs in the graph of $y = f(x)$.

Example 1

In the previous lesson, we studied a simple type of instruction that computers perform called a for-next loop. Another simple type of instruction is an *if-then statement*. Below is example code of a program that tests for and prints "True" when $x + 2 = 4$; otherwise it prints "False."

```
Declare x integer
For all x from 1 to 4
    If x + 2 = 4 then
        Print True
    else
        Print False
    End if
Next x
```

The output of this program code is

False
True
False
False

Notice that the if-then statement in the code above is really just testing whether each number in the loop is in the solution set.

Example 2

Perform the instructions in the following programming code as if you were a computer and your paper were the computer screen.

```
Declare x integer
Initialize G as {}
For all x from 0 to 4
    If x² − 4x + 5 = 2 then
        Append x to G
    else
        Do NOT append x to G
    End if
Next x
Print G
```

Output: $\{1, 3\}$

Discussion

Compare the for-next/if-then code above to the following set-builder notation we used to describe solution sets in Module 1:

$$\{x \text{ integer} \mid 0 \le x \le 4 \text{ and } x^2 - 4x + 5 = 2\}.$$

Check to see that the set-builder notation also generates the set $\{1, 3\}$. *Whenever you see set-builder notation to describe a set, a powerful way to interpret that notation is to think of the set as being generated by a program like the for-next or if-then code above.*

Exploratory Challenge 1

Next we write code that generates a graph of a *two-variable equation* $y = x(x - 2)(x + 2)$ for x in $\{-2, -1, 0, 1, 2\}$ and y in $\{-3, 0, 3\}$. The solution set of this equation is generated by testing each ordered pair (x, y) in the set,

$$\{(-2, -3), (-2, 0), (-2, 3), (-1, -3), (-1, 0), (-1, 3), \dots, (2, -3), (2, 0), (2, 3)\},$$

to see if it is a solution to the equation $y = x(x - 2)(x + 2)$. Then the graph is just the plot of solutions in the Cartesian plane. We can instruct a computer to find these points and plot them using the following program.

> **Declare** x and y **integers**
> **Initialize** G **as** {}
> **For all** x **in** $\{-2, -1, 0, 1, 2\}$
>   **For all** y **in** $\{-3, 0, 3\}$
>   **If** $y = x(x - 2)(x + 2)$ **then**
>   **Append** (x, y) **to** G
>   **else**
>   **Do NOT append** (x, y) **to** G
>   **End if**
>   **Next** y
> **Next** x
> **Print** G
> **Plot** G

Tests whether (x, y) is a solution.
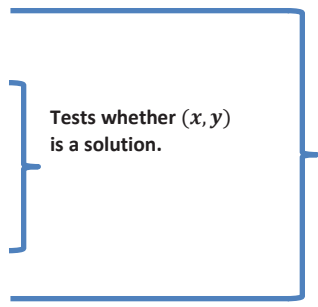
Loops through each y for $x = -2$, then for $x = -1$, and so on (see arrows in table below).

a. Use the table below to record the decisions a computer would make when following the program instructions above. Fill in each cell with "Yes" or "No" depending on whether the ordered pair (x, y) would be appended or not. (The step where $x = -2$ has been done for you.)

	$x = -2$	$x = -1$	$x = 0$	$x = 1$	$x = 2$
$y = 3$	No				
$y = 0$	Yes				
$y = -3$	No				

b. What would be the output to the Print G command? (The first ordered pair is listed for you.)

Output:
{ $(-2, 0)$, _____, _____, _____, _____ }

c.   Plot the solution set G in the Cartesian plane. (The first ordered pair in G has been plotted for you.)
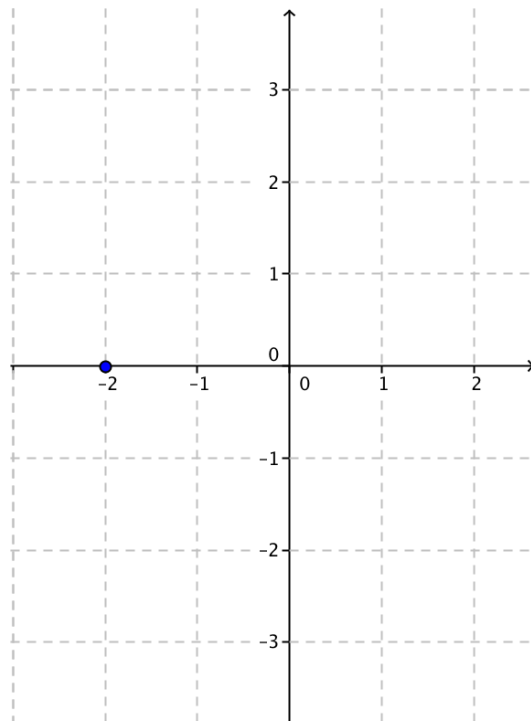

Exploratory Challenge 2

The program code in Exercise 3 is a way to imagine how set-builder notation generates solution sets and figures in the plane. Given a function $f(x) = x(x - 2)(x - 3)$ with domain and range all real numbers, a slight modification of the program code above can be used to generate the graph of the equation $y = f(x)$:

$$\{(x, y) \,|\, x \text{ real } \text{ and } y = f(x)\}.$$

Even though the code below cannot be run on a computer, students can run the following thought code in their minds.

```
Declare x and y real
Let f(x) = x(x − 2)(x + 2)
Initialize G as {}
For all x in the real numbers
        For all y in the real numbers
                If y = f(x) then
                        Append (x, y) to G
                else
                        Do NOT append (x, y) to G
                End if
        Next y
Next x
Plot G
```
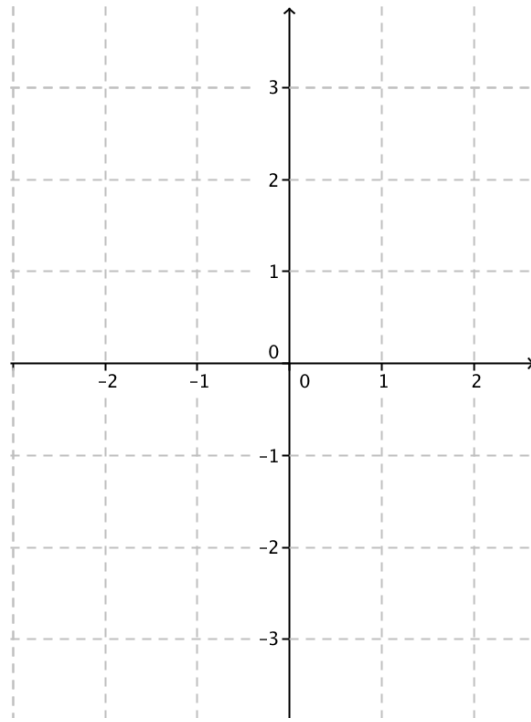
Tests whether (x, y) is a solution to $y = x(x - 2)(x + 2)$.
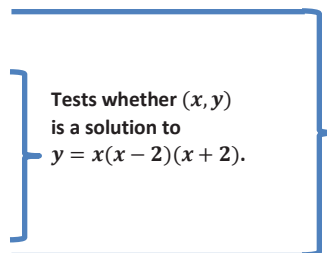
For each x-value, the code loops through all y-values.

a.   Plot G on the Cartesian plane (the figure drawn is called the graph of $y = f(x)$).

b.   Describe how the thought code is similar to the set-builder notation $\{(x, y) \mid x \text{ real and } y = f(x)\}$.

c.   A *relative maximum* for the function f occurs at the x-coordinate of $\left(-\frac{2}{3}\sqrt{3}, \frac{16}{9}\sqrt{3}\right)$. Substitute this point into the equation $y = x(x^2 - 4)$ to check that it is a solution to $y = f(x)$, and then plot the point on your graph.

d. A *relative minimum* for the function f occurs at the x-coordinate of $\left(\frac{2}{3}\sqrt{3}, -\frac{16}{9}\sqrt{3}\right)$. A similar calculation as you did above shows that this point is also a solution to $y = f(x)$. Plot this point on your graph.

e. Look at your graph. On what interval(s) is the function f decreasing?

f. Look at your graph. On what interval(s) is the function f increasing?

Lesson Summary

- **GRAPH OF $y = f(x)$:** Given a function f whose domain D, and the range are subsets of the real numbers, the *graph of $y = f(x)$* is the set of ordered pairs (x, y) in the Cartesian plane given by

$$\{(x, y) \mid x \in D \text{ and } y = f(x)\}.$$

When we write $\{(x, y) \mid y = f(x)\}$ for the graph of $y = f(x)$, it is understood that the domain is the largest set of real numbers for which the function f is defined.

- The graph of f is the same as the graph of the equation $y = f(x)$.

- **INCREASING/DECREASING:** Given a function f whose domain and range are subsets of the real numbers, and I is an interval contained within the domain, the function is called *increasing on the interval I* if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$

It is called *decreasing on the interval I* if

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$

Problem Set

1. Perform the instructions in the following programming code as if you were a computer and your paper were the computer screen.

```
Declare x integer
For all x from 1 to 6
    If x² − 2 = 7 then
        Print True
    else
        Print False
    End if
Next x
```

2. Answer the following questions about the computer programming code.

> **Declare** x **integer**
> **Initialize** G **as {}**
> **For all** x **from** -3 **to** 3
> > **If** $2^x + 2^{-x} = \frac{17}{4}$ **then**
> > > **Append** x **to** G
> >
> > **else**
> > > **Do NOT append** x **to** G
> >
> > **End if**
>
> **Next** x
> **Print** G

a. Perform the instructions in the programming code as if you were a computer and your paper were the computer screen.

b. Write a description of the set G using set-builder notation.

3. Answer the following questions about the computer programming code.

> **Declare** x **and** y **integers**
> **Initialize** G **as {}**
> **For all** x **in** $\{0, 1, 2, 3\}$
> > **For all** y **in** $\{0, 1, 2, 3\}$
> > > **If** $y = \sqrt{4 + 20x - 19x^2 + 4x^3}$ **then**
> > > > **Append** (x, y) **to** G
> > >
> > > **else**
> > > > **Do NOT append** (x, y) **to** G
> > >
> > > **End if**
> >
> > **Next** y
>
> **Next** x
> **Plot** G

a. Use the table below to record the decisions a computer would make when following the program instructions above. Fill in each cell with "Yes" or "No" depending on whether the ordered pair (x, y) would be appended or not.

	$x = 0$	$x = 1$	$x = 2$	$x = 3$
$y = 3$				
$y = 2$				
$y = 1$				
$y = 0$				

b. Plot the set G in the Cartesian plane.
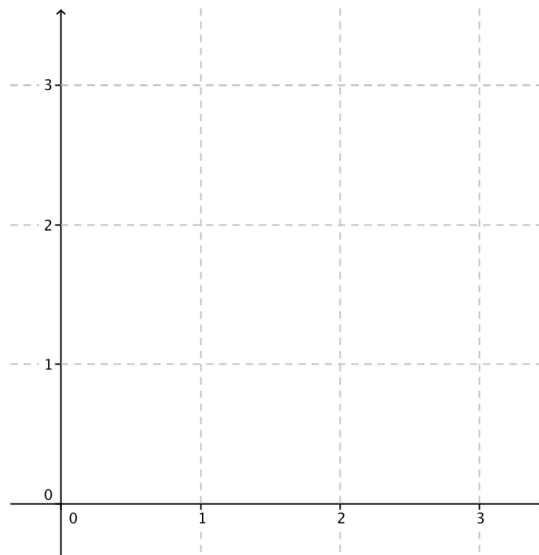
4. Answer the following questions about the thought code.

Declare x and y real
Let $f(x) = -2x + 8$
Initialize G as {}
For all x in the real numbers
    For all y in the real numbers
        If $y = f(x)$ then
            Append (x, y) to G
        else
            Do NOT append (x, y) to G
        End if
    Next y
Next x
Plot G

a. What is the domain of the function $f(x) = -2x + 8$?

b. What is the range of the function $f(x) = -2x + 8$?

c. Write the set G generated by the thought code in set-builder notation.

d. Plot the set G to obtain the graph of the function $f(x) = -2x + 8$.

e. The function $f(x) = -2x + 8$ is clearly a decreasing function on the domain of the real numbers. Show that the function satisfies the definition of decreasing for the points 8 and 10 on the number line; that is, show that since $8 < 10$, then $f(8) > f(10)$.

5. Sketch the graph of the functions defined by the following formulas, and write the graph of $y = f(x)$ as a set using set-builder notation. (Hint: For each function below, you can assume the domain is all real numbers.)

   a. $f(x) = -\frac{1}{2}x + 6$

   b. $f(x) = x^2 + 3$

   c. $f(x) = x^2 - 5x + 6$

   d. $f(x) = x^3 - x$

   e. $f(x) = -x^2 + x - 1$

   f. $f(x) = (x - 3)^2 + 2$

   g. $f(x) = x^3 - 2x^2 + 3$

6. Answer the following questions about the set:

$$\{(x, y) \mid 0 \leq x \leq 2 \text{ and } y = 9 - 4x^2\}.$$

   a. The equation can be rewritten in the form $y = f(x)$ where $f(x) = 9 - 4x^2$. What are the domain and range of the function f specified by the set?

      i. Domain:

      ii. Range:

   b. Write thought code such as that in Problem 4 that will generate and then plot the set.

7. Answer the following about the graph of a function below.

   a. Which points (A, B, C, or D) are relative maxima?

   b. Which points (A, B, C, or D) are relative minima?

   c. Name any interval where the function is increasing.

   d. Name any interval where the function is decreasing.